◎ Gauss – Jordan Elimination

Given A, we want to find its inverse A^{-1}.

$$AA^{-1} = I$$

$$A\begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix}$$

Gauss

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Jordan

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$$\therefore\ \underline{x}_1 = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \quad \underline{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \quad \underline{x}_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

In another viewpoint,

$[\,A \mid I\,] \Rightarrow [\,I \mid A^{-1}\,]$ by performing elementary row operations

because $A^{-1}[\,A \mid I\,] = [\,I \mid A^{-1}\,]$

Def An $n \times n$ matrix is <u>nonsingular</u> if it has a full set of n (nonzero) pivots.

Claim A matrix is invertible if and only if it is nonsingular.
(iff)

Proof " $\Leftarrow$ "

Suppose A is nonsingular, i.e., it has a fullset of n pivots.
Then by Gauss–Jordan elimination, we can find a matrix B such that
$AB = I$ (since $A\underline{x_i} = \underline{e_i}$ is solvable for all $i = 1, 2, \ldots\ldots, n$)
right inverse

On the other hand, Gauss–Jordan elimination is really a sequence of multiplications by elementary matrices on the left $(D^{-1} \cdots E \cdots \underline{P} \cdots E)A = I$

where E_{ij} : to subtract a multiple l_{ij} of row j from row i.
$\underline{P}_{ij}$: to exchange rows i and j.
D^{-1}: to divide all rows by their pivots.
That is, there is a matrix G such that $GA = I$
Therefore, $B = G = A^{-1}$ and A is invertible.

" $\Rightarrow$ " If A does not have n pivots, (A is singular) elimination will lead to a zero row,
i.e., there is an invertible M such that a row of MA is zero.
If $AG = I$ is possible, then $MAG = M$

$$\underbrace{(MA)}_{\text{has a zero row}}\, G = \underbrace{M}_{\text{has a zero row}}$$

Hence M must have a zero row, which reaches a contridiction since M is invertible. Therefore, A is not invertible. ■

◎ Elimination = Factorization : $A = LU$

$$\begin{array}{c} \\ \times -2 \\ \\ \end{array}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{c} \\ \times 1 \\ \\ \end{array}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{c} \\ \times 1 \\ \\ \end{array}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(E_{32}E_{31}E_{21})A = U \rightarrow \text{upper triangular}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$$(E_{32}E_{31}E_{21})A = U$$

$$\Rightarrow A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$$

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

In general,

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$= L \quad \text{lower-triangular}$$

Note $E_{32} E_{31} E_{21} \neq \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{32} & 1 \end{bmatrix}$

$\therefore A = LU$ if no row exchanges are required

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

We can further split U into

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pivots}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A = LDU$ if no row exchanges are required

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

where L is a lower-triangular matrix with 1's on the diagonal.
(which records the steps of elimination)
D is a diagonal matrix with pivots on the diagonal.
U is an upper-triangular atrix with 1's on the diagonal.

<u>Claim</u> If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, where the L's are lower-triangular with unit diagonal, the U's are upper-triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$.

<u>Proof</u> Exercise.